The General Definition of the Complex Monge-Ampère Operator on Compact Kähler Manifolds

Yang Xing

Abstract. We introduce a wide subclass $\mathcal{F}(X,\omega)$ of quasi-plurisubharmonic functions in a compact Kähler manifold, on which the complex Monge-Ampère operator is well-defined and the convergence theorem is valid. We also prove that $\mathcal{F}(X,\omega)$ is a convex cone and includes all quasi-plurisubharmonic functions which are in the Cegrell class.

1. Introduction

Let X be a compact connected Kähler manifold of dimension n, equipped with the fundamental form ω given in local coordinates by $\omega = \frac{i}{2}\sum_{\alpha,\beta} g_{\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta$, where $(g_{\alpha\bar\beta})$ is a positive definite Hermitian matrix and $d\omega = 0$. The smooth volume form associated to this Kähler metric is the nth wedge product ω^n. Denote by $PSH(X,\omega)$ the set of upper semi-continuous functions $u : X \to \mathbb{R}\cup\{-\infty\}$ such that u is integrable in X with respect to the volume form ω^n and $\omega_u := \omega + dd^c u \geq 0$ on X, where $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)$. These functions are called quasi-plurisubharmonic functions (quasi-psh for short) and play an important role in the study of positive closed currents in X, see Demailly's paper [D1]. A quasi-psh function is locally the difference of a plurisubharmonic function and a smooth function. Therefore, many properties of plurisubharmonic functions hold also for quasi-psh functions. Following Bedford and Taylor [BT2], the complex Monge-Ampère operator $(\omega + dd^c)^n$ is locally and hence globally well defined for all bounded quasi-psh functions in X. Some important results of the complex Monge-Ampère operator for bounded quasi-psh functions have been obtained by Kolodziej [KO1-2] and Blocki [BL1]. It is also known that the complex Monge-Ampère operator does not work well for all unbounded quasi-psh functions. Otherwise, we shall lose some of the essential properties that the complex Monge-Ampère operator should have, see Kiselman's paper

Yang Xing
Department of Mathematics, University of Umeå, S-901 87 Umeå, Sweden
E-mail address: Yang.Xing@mathdept.umu.se

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[KI] or Bedford's survey [B]. In a bounded domain of $\mathbb{C}^n$ one usually needs certain assumptions on values of functions near the boundary of the domain to define complex Monge-Ampère measures of unbounded plurisubahrmonic functions, see the Cegrell class [C1-2] where Cegrell introduced the largest subclass $\mathcal{E}(\Omega)$ of plurisuhharmonic functions in a bounded hyperconvex domain Ω for which the complex Monge-Ampère operator is well-defined and the monotone convergence theorem is valid. However, such a technique does not seem to work for quasi-psh functions in a compact Kähler manifold because we lose boundary. On the other hand, it was already observed by Bedford and Taylor [BT1] that for each quasi-psh function u the complex Monge-Ampère measure $\omega_u^n := (\omega + dd^c u)^n$ is well defined on its non-polar subset $\{u > -\infty\}$. The complex Monge-Ampère measures ω_u^n concentrating on $\{u > -\infty\}$ were studied by Guedj and Zeriahi [GZ]. In [X3] we obtained several convergence theorems for complex Monge-Ampère measures without mass on pluripolar sets. In this paper we introduce a quite large subclass $\mathcal{F}(X,\omega)$ of quasi-psh functions on which images of the complex Monge-Ampère operator are well-defined positive measures and may have positive masses on pluripolar sets. We prove that the set $\mathcal{F}(X,\omega)$ is a convex cone and includes all quasi-psh functions which are in the Cegrell class. Our main result is the following convergence theorem of the complex Monge-Ampère operator in $\mathcal{F}(X,\omega)$.

Theorem 5.(Convergence Theorem) *Let $0 \le p < \infty$. Suppose that $u_0 \in \mathcal{F}(X,\omega)$ and that $g \in PSH(X,\omega) \cap L^\infty(X)$ is nonpositive. If u_j, $u \in \mathcal{F}(X,\omega)$ are such that $u_j \to u$ in Cap_ω on X and $u_j \ge u_0$, then $(-g)^p\, \omega_{u_j}^n \to (-g)^p\, \omega_u^n$ weakly in X.*

As a direct consequence we have

Corollary 5. *Let $0 \le p < \infty$ and $0 \ge g \in PSH(X,\omega) \cap L^\infty(X)$. If u_j, $u \in \mathcal{F}(X,\omega)$ are such that $u_j \searrow u$ or $u_j \nearrow u$ in X, then $(-g)^p\, \omega_{u_j}^n \to (-g)^p\, \omega_u^n$ weakly in X.*

For bounded quasi-psh functions, Corollary 5 is a slightly stronger version of the well-known monotone convergence theorem due to Bedford and Taylor [BT2].

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2. The class $\mathcal{F}(X,\omega)$

In this section we first introduce the subclass $\mathcal{F}(X,\omega)$ of quasi-psh functions, on which images of the complex Monge-Ampère operator are finite positive measures in X. We obtain some characterizations of functions in $\mathcal{F}(X,\omega)$. Finally, we prove that $\mathcal{F}(X,\omega)$ is a star-shaped and convex set.

Recall that the Monge-Ampère capacity Cap_ω associated to the Kähler form ω is defined by

$$Cap_\omega(E) = \sup\Big\{ \int_E \omega_u^n;\ u \in PSH(X,\omega) \text{ and } -1 \le u \le 0 \Big\},$$

for any Borel set E in X. The capacity Cap_ω is introduced by Kolodziej [KO1] and is comparable to the relative Monge-Ampère capacity of Bedford and Taylor [BT2], and hence vanishes exactly on pluripolar sets of X. Recall also that a sequence μ_j of positive Borel measures is said to be uniformly absolutely continuous with respect to Cap_ω on X, or we write that $\mu_j \ll Cap_\omega$ on X uniformly for all j, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu_j(E) < \varepsilon$ for all j and Borel sets $E \subset X$ with $Cap_\omega(E) < \delta$. Denote by $PSH^{-1}(X, \omega)$ the subset of functions u in $PSH(X, \omega)$ with $\max\limits_X u \leq -1$. Given a function $u \in PSH^{-1}(X, \omega)$, we define the measure $(-u)\, \omega_u^{n-1} \wedge \omega$ in X which is zero in $\{u = -\infty\}$ and

$$\int\limits_E (-u)\, \omega_u^{n-1} \wedge \omega = \lim_{j\to\infty} \int\limits_{E\cap\{u>-j\}} \big(-\max(u, -j)\big)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega$$

for all $k \geq 1$ and $E \subset \{u > -k\}$. In a completely similar way, we define the measure $\omega_u^{n-1} \wedge \omega := \chi_{\{u>-\infty\}}\, \omega_u^{n-1} \wedge \omega$, where $\chi_{\{u>-\infty\}}$ is the characteristic function of the set $\{u > -\infty\}$. It is worth to point out that in general neither the measure $(-u)\, \omega_u^{n-1} \wedge \omega$ nor $\omega_u^{n-1} \wedge \omega$ is locally finite in X. However, we have the following result.

Proposition 1. *Let $u \in PSH^{-1}(X, \omega)$. Suppose that*

$$-\max(u, -j)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega \ll Cap_\omega \quad \text{on } X \text{ uniformly for all } j = 1, 2, \ldots.$$

Then the following statements hold.

(1) $(-u)\, \omega_u^{n-1} \wedge \omega$ and $\omega_u^{n-1} \wedge \omega$ are finite positive measures in X;
(2) $\max(u, -j)\, \omega^{n-1}_{\max(u,-j)} \to u\, \omega_u^{n-1}$ and $\omega^{n-1}_{\max(u,-j)} \to \omega_u^{n-1}$ as currents as $j \to \infty$;
(3) $(-u)\, \omega_u^{n-1} \wedge \omega \ll Cap_\omega$ on X.

Proof. Since $\int_X (-u)\, \omega_u^{n-1} \wedge \omega = \lim_{k\to\infty} \lim_{j\to\infty} \int_{u>-k} \big(-\max(u, -j)\big)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega \leq \sup\limits_j \int_X \big(-\max(u, -j)\big)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega < \infty$, we obtain that $(-u)\, \omega_u^{n-1} \wedge \omega$ is a finite positive measure and so is $\omega_u^{n-1} \wedge \omega$. Write $\max(u, -j)\, \omega^{n-1}_{\max(u,-j)} = \chi_{\{u\leq -j\}} \max(u, -j)\, \omega^{n-1}_{\max(u,-j)}$ $+\chi_{\{u>-j\}} \max(u, -j)\, \omega^{n-1}_{\max(u,-j)}$, where the first term on the right hand side tends to zero and the second one tends to $u\, \omega_u^{n-1}$ as $j \to \infty$. Similarly, we get that $\omega^{n-1}_{\max(u,-j)} \to \omega_u^{n-1}$ as $j \to \infty$. Moreover, for any $E \subset X$ with $Cap_\omega(E) \neq 0$ we can take an open set G in X such that $E \subset G$ and $Cap_\omega(G) \leq 2\, Cap_\omega(E)$. Then $\int_E (-u)\, \omega_u^{n-1} \wedge \omega \leq \int_G ((-u)\, \omega_u^{n-1} \wedge \omega \leq \limsup\limits_{j\to\infty} \int_G \big(-\max(u, -j)\big)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega$, which implies that $(-u)\, \omega_u^{n-1} \wedge \omega \ll Cap_\omega$ on X and the proof of Proposition 1 is complete.

Let $\mathcal{F}(X, \omega)$ be the subset of functions in $PSH^{-1}(X, \omega)$ which satisfy the hypotheses of Proposition 1. The complex Monge-Ampère measure ω_u^n of a function u in $\mathcal{F}(X, \omega)$ is defined by the sum

$$\omega_u^n := \omega \wedge \omega_u^{n-1} + dd^c(u\, \omega_u^{n-1}),$$

where the currents $u\, \omega_u^{n-1}$ and ω_u^{n-1} are the limits of two sequences $\max(u, -j)\, \omega^{n-1}_{\max(u,-j)}$ and $\omega^{n-1}_{\max(u,-j)}$ respectively. Locally using the inequality $\big(\omega + dd^c(\phi + u)\big)^n \geq n\, \omega_u^{n-1} \wedge$

ω, where $\omega = dd^c\phi$, we can easily see that $(-u)\,\omega_u^{n-1}\wedge\omega \ll Cap_\omega$ in X for any $u \in PSH^{-1}(X,\omega)\cap L^\infty(X)$, where $L^\infty(X)$ denotes the set of bounded functions in X. Hence for bounded quasi-psh functions, our definition of the complex Monge-Ampère operator coincides with Bedford's and Taylor's definition given in [BT2]. Denote by $L^1(X,\mu)$ the set of integrable functions in X with respect to the positive measure μ. Now we give a characterization of functions in $\mathcal{F}(X,\omega)$.

Theorem 1. *Let $u \in PSH^{-1}(X,\omega)$. Then $u \in \mathcal{F}(X,\omega)$ if and only if*

$$u \in L^1(X,\,\omega_u^{n-1}\wedge\omega),$$

where $\omega_u^{n-1} := \lim_{j\to\infty}\omega_{\max(u,-j)}^{n-1}$ as currents and $\omega_{\max(u,-j)}^{n-1}\wedge\omega \ll Cap_\omega$ on X uniformly for $j = 1,2,\ldots$.

Proof. We prove first the " only if " part. Assume that $u \in \mathcal{F}(X,\omega)$. By Proposition 1 we have that $\omega_{\max(u,-j)}^{n-1}\wedge\omega \le \big(-\max(u,-j)\big)\,\omega_{\max(u,-j)}^{n-1}\wedge\omega \ll Cap_\omega$ on X uniformly for all j, and $\omega_{\max(u,-j)}^{n-1} \longrightarrow \omega_u^{n-1}$. Hence, by the lower semi-continuity of $-u$, we get that $\int_X\big(-\max(u,-t)\big)\,\omega_u^{n-1}\wedge\omega \le \limsup_{j\to\infty}\int_X\big(-\max(u,-j)\big)\,\omega_{\max(u,-j)}^{n-1}\wedge\omega < \infty$ for all $t \ge 1$. Thus, we have $u \in L^1(X,\,\omega_u^{n-1}\wedge\omega)$. Now we prove the " if " part. Observe that for any $k > 1$, by Proposition 4.2 in [BT1] we get $\chi_{\{u>-k\}}\,\omega_u^{n-1}\wedge\omega = \lim_{j\to\infty}\chi_{\{u>-k\}}\omega_{\max(u,-j)}^{n-1}\wedge\omega = \lim_{j\to\infty}\chi_{\{\max(u,-k)>-k\}}\omega_{\max(u,-j)}^{n-1}\wedge\omega = \lim_{j\to\infty}\chi_{\{\max(u,-k)>-k\}}\,\omega_{\max(u,-j,-k)}^{n-1}\wedge\omega = \chi_{\{u>-k\}}\,\omega_{\max(u,-k)}^{n-1}\wedge\omega$. Hence, for any Borel set $E \subset X$ and $k > 1$, we have that $\int_E\omega_u^{n-1}\wedge\omega \le \int_{u<-k+1}\omega_u^{n-1}\wedge\omega + \int_{E\cap\{u>-k\}}\omega_{\max(u,-k)}^{n-1}\wedge\omega \le \limsup_{j\to\infty}\int_{u<-k+1}\omega_{\max(u,-j)}^{n-1}\wedge\omega + \int_E\omega_{\max(u,-k)}^{n-1}\wedge\omega$, where we have used that the set $\{u < -k+1\}$ is open. Since $\omega_{\max(u,-j)}^{n-1}\wedge\omega \ll Cap_\omega$ on X uniformly for j, we have $\omega_u^{n-1}\wedge\omega \ll Cap_\omega$ on X. It then follows from $u \in L^1(X,\,\omega_u^{n-1}\wedge\omega)$ that $(-u)\,\omega_u^{n-1}\wedge\omega \ll Cap_\omega$ on X. For any $j \ge k_1 > 1$ we get

$$\int\limits_{u\le -k_1}\big(-\max(u,-j)\big)\,\omega_{\max(u,-j)}^{n-1}\wedge\omega \le j\int\limits_{u\le -j}\omega_{\max(u,-j)}^{n-1}\wedge\omega + \int\limits_{-j<u\le -k_1}(-u)\,\omega_u^{n-1}\wedge\omega$$

$$= j\int\limits_X\omega^n - j\int\limits_{u>-j}\omega_{\max(u,-j)}^{n-1}\wedge\omega + \int\limits_{-j<u\le -k_1}(-u)\,\omega_u^{n-1}\wedge\omega$$

$$\le j\int\limits_X\omega^n - j\int\limits_{u>-j}\omega_u^{n-1}\wedge\omega + \int\limits_{u\le -k_1}(-u)\,\omega_u^{n-1}\wedge\omega \le \int\limits_{\{u\le -j\}\cup\{u\le -k_1\}}(-u)\,\omega_u^{n-1}\wedge\omega.$$

Hence, for any Borel set $E_1 \subset X$ and $j \ge k_1 > 1$, we have $\int_{E_1}\big(-\max(u,-j)\big)\,\omega_{\max(u,-j)}^{n-1}\wedge\omega \le \int_{\{u\le -j\}\cup\{u\le -k_1\}}(-u)\,\omega_u^{n-1}\wedge\omega + k_1\int_{E_1\cap\{u>-k_1\}}\omega_{\max(u,-j)}^{n-1}\wedge\omega := A_{k_1,j} + B_{k_1,j}$. Given $\varepsilon > 0$ take $k_\varepsilon > 1$ and $j_\varepsilon > 1$ such that $A_{k_\varepsilon,j} \le \varepsilon$ for all $j \ge j_\varepsilon$. Since $\omega_{\max(u,-j)}^{n-1}\wedge\omega \ll Cap_\omega$ on X uniformly for all j, there exists $\delta > 0$ such that $(j_\varepsilon +$

k_ε) $\int_{E_1} \omega^{n-1}_{\max(u,-j)} \wedge \omega \leq \varepsilon$ for all j and $E_1 \subset X$ with $Cap_\omega(E_1) \leq \delta$. Therefore, we have proved that $\int\limits_{E_1} \big(-\max(u,-j)\big)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega \leq 2\,\varepsilon$ holds for all j and $E_1 \subset X$ with $Cap_\omega(E_1) \leq \delta$. So $u \in \mathcal{F}(X,\omega)$ and the proof of Theorem 1 is complete.

Suppose that Ω is a hyperconvex subset in $\mathbb{C}^n$. Cegrell [C2] introduced the largest subclass $\mathcal{E}(\Omega)$ of plurisuhharmonic functions in Ω, for which the complex Monge-Ampère operator is well-defined and the monotone convergence theorem is valid. Our next theorem says that $\mathcal{F}(X,\omega)$ includes all quasi-psh functions which are in the Cegrell class. Recall that a negative plurisubharmonic function u in Ω is said to belong to $\mathcal{E}(\Omega)$ if for each $z_0 \in \Omega$ there exist a neighborhood U_{z_0} of z_0 and a decreasing sequence u_j of bounded plurisubharmonic functions in Ω, vanishing on the boundary $\partial\Omega$, such that $u_j \searrow u$ on U_{z_0} and $\sup\limits_j \int_\Omega (dd^c u_j)^n < \infty$. Blocki proved in [BL2] that it is a local property to belong to $\mathcal{E}(\Omega)$, that is, if $\Omega = \cup_j \Omega_j$ then $u \in \mathcal{E}(\Omega)$ if and only if $u\big|_{\Omega_j} \in \mathcal{E}(\Omega_j)$ for each j. We call u in $PSH^{-1}(X,\omega)$ for a Cegrell function in X if there exists a finite covering $\{B_s\}_1^m$ of X with hyperconvex subsets B_s such that $\phi_s + u \in \mathcal{E}(B_s)$ for all s, where ϕ_s is a local Kähler potential defined in a neighborhood of the closure of B_s, i.e. $\omega = dd^c\phi_s$ on $B_s = \{\phi_s < 0\}$. Now we prove

Theorem 2. *If u is a Cegrell function in X then $u \in \mathcal{F}(X,\omega)$.*

Proof. Take a new finite open covering $\{B'_s\}_1^m$ of X such that $B'_s \subset\subset B_s$ for all s. By [C2] there exists a decreasing sequence u_j^s of bounded plurisubharmonic functions in B_s, vanishing on ∂B_s, such that $u_j^s \searrow \phi_s + u$ on B'_s and $\sup\limits_j \int_{B_s} (dd^c u_j^s)^n < \infty$. Since Cap_ω is comparable to the relative Monge-Ampère capacity of Bedford and Taylor, see [KO2][BT2], by Lemma 6 in [X2] we get that $-\max(u,-j)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega \leq \big(-\phi_s - \max(u,-j)\big)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega \ll Cap_\omega$ uniformly for all j on each B'_s and hence on X. Therefore, $u \in \mathcal{F}(X,\omega)$ and the proof is complete.

Recall that a sequence u_j of functions in X is said to be convergent to a function u in Cap_ω on X if for any $\delta > 0$ we have

$$\lim_{j\to\infty} Cap_\omega\big(\{z \in X; |u_j(z) - u(z)| > \delta\}\big) = 0.$$

For a uniformly bounded sequence in $PSH(X,\omega)$, the convergence in capacity implies weak convergence of the complex Monge-Ampère measures [X1]. Now we prove that the set $\mathcal{F}(X,\omega)$ is a convex cone. First, we need a lemma.

Lemma 1. *Let u, $v \in \mathcal{F}(X,\omega)$. Then*

$$\int\limits_{u<v} (v-u)\, \omega_v^{n-1} \wedge \omega \leq \int\limits_{u<v} (v-u)\, \omega_u^{n-1} \wedge \omega.$$

If furthermore u and v are bounded, then for all integers $0 \le l \le n-1$ we have

$$\int_{u<v} (v-u)\,\omega_v^l \wedge \omega_u^{n-1-l} \wedge \omega \le \int_{u<v} (v-u)\,\omega_u^{n-1} \wedge \omega.$$

Proof. We only prove the first inequality since the proof of the second one is similar. Assume first that u and v are bounded in X. By [D1] there exist a constant $A>1$ and two sequences $u_j,\, v_k \in PSH(X, A\omega)\cap C^\infty(X)$ such that $u_j \searrow u$ and $v_k \searrow v$ in X. Given $\varepsilon>0$. Assume first that $\{u_j<v_k\} \ne X$. Then $\max(v_k, u_j+\varepsilon) = u_j+\varepsilon$ near the boundary of the set $\{u_j<v_k\}$. Take a smooth subset E_ε such that $\{u_j+\varepsilon<v_k\} \subset\subset E_\varepsilon \subset\subset \{u_j<v_k\}$, and write $T = \sum_{l=0}^{n-2} \omega_u^l \wedge \omega_v^{n-2-l} \wedge \omega$. Using Stokes theorem we get

$$\int_{u_j<v_k} \big(\max(v_k, u_j+\varepsilon) - u_j - \varepsilon\big)\,\big((A\omega + dd^c u_j) - (A\omega + dd^c \max(v_k, u_j+\varepsilon))\big) \wedge T$$

$$= \int_{E_\varepsilon} d\big(\max(v_k, u_j+\varepsilon) - u_j\big) \wedge d^c\big(\max(v_k, u_j+\varepsilon) - u_j\big) \wedge T \ge 0,$$

which holds even when $\{u_j<v_k\} = X$. Hence we obtain $\int_{u_j<v_k} \big(\max(v_k, u_j+\varepsilon) - u_j\big)\,(A\omega + dd^c u_j) \wedge T \ge \int_{u_j<v_k} \big(\max(v_k, u_j+\varepsilon) - u_j - \varepsilon\big)\,(A\omega + dd^c \max(v_k, u_j+\varepsilon)) \wedge T \ge \int_{u_j<v_k} (v_k - u_j)\,(A\omega + dd^c \max(v_k, u_j+\varepsilon)) \wedge T - \varepsilon\, A \int_X \omega^n$. It turns out from the monotone convergence theorem in [BT2] that $(v_k-u_j)\,(A\omega + dd^c \max(v_k, u_j+\varepsilon)) \wedge T \longrightarrow (v_k-u_j)\,(A\omega + dd^c v_k)) \wedge T$ weakly in the open set $\{u_j<v_k\}$ as $\varepsilon \searrow 0$. Letting $\varepsilon \searrow 0$ and applying Lebesgue monotone convergence theorem we obtain the inequality $\int_{u_j<v_k} (v_k-u_j)\,(A\omega + dd^c v_k) \wedge T \le \int_{u_j<v_k} (v_k-u_j)\,(A\omega + dd^c u_j) \wedge T$. Therefore, we have $\int_{u_j<v} (v-u_j)\,(A\omega + dd^c v_k) \wedge T \le \int_{u<v_k} (v_k-u)\,(A\omega + dd^c u_j) \wedge T$. On the other hand, we have that $u_j,\, v_k$ are uniformly bounded, $u_j \to u$ in Cap_ω and $v_k \to v$ in Cap_ω on X. So for any $\delta>0$ the inequality $\int_{u<v} (v-u_j)\,(A\omega + dd^c v_k) \wedge T \le \int_{u\le v} (v_k-u)\,(A\omega + dd^c u_j) \wedge T + \delta$ holds for all j, k large enough. Then by the quasicontinuity of quasi-psh functions, we can assume without loss of generality that $\{u<v\}$ is open and $\{u\le v\}$ is closed. It turns out from the proof of Theorem 1 in [X1] that $(v-u_j)\,(A\omega + dd^c v_k) \wedge T \longrightarrow (v-u_j)\,(A\omega + dd^c u) \wedge T$ as $k\to\infty$ and $(v-u)\,(A\omega + dd^c u_j) \wedge T \longrightarrow (v-u)\,(A\omega + dd^c v) \wedge T$ as $j\to\infty$ weakly in X. Letting $k\to\infty$ and then $j\to\infty$, we obtain $\int_{u<v} (v-u)\,(A\omega + dd^c v) \wedge T \le \int_{u\le v} (v-u)\,(A\omega + dd^c u) \wedge T + \delta$. Applying $t\,v$ instead of v for $A>t>1$ in the last inequality and then letting $t \searrow 1$, $\delta \searrow 0$ we get $\int_{u<v} (v-u)\,(A\omega + dd^c v) \wedge T \le \int_{u<v} (v-u)\,(A\omega + dd^c u) \wedge T$, which yields that $\int_{u<v} (v-u)\,\omega_v^{n-1} \wedge \omega \le \int_{u<v} (v-u)\,\omega_u^{n-1} \wedge \omega$ for all bounded quasi-psh functions u and v.

Now, for $u,\, v \in \mathcal{F}(X,\omega)$, we have $\int_{\max(u,-j)<\max(v,-k)} \big(\max(v,-k) - \max(u,-j)\big)\, \omega_{\max(v,-k)}^{n-1} \wedge \omega \le \int_{\max(u,-j)<\max(v,-k)} \big(\max(v,-k) - \max(u,-j)\big)\, \omega_{\max(u,-j)}^{n-1} \wedge \omega$. Letting $k\to\infty$, by the definition of $\omega_v^{n-1} \wedge \omega$ we get $\int_{\max(u,-j)<v} \big(v - \max(u,-j)\big)\, \omega_v^{n-1} \wedge \omega \le$

$\int_{\max(u,-j)<v}\big(v-\max(u,-j)\big)\,\omega_{\max(u,-j)}^{n-1}\wedge\omega$, which by Fatou lemma implies that

$$\int\limits_{u<v}\big(v-u\big)\,\omega_v^{n-1}\wedge\omega \leq \liminf_{j\to\infty}\int\limits_{\max(u,-j)<v}\big(v-\max(u,-j)\big)\,\omega_{\max(u,-j)}^{n-1}\wedge\omega$$

$$\leq \liminf_{j\to\infty}\int\limits_{u<v}\big(\max(v,-j)-\max(u,-j)\big)\,\omega_{\max(u,-j)}^{n-1}\wedge\omega$$

$$\leq \limsup_{j\to\infty}\int\limits_{-s<u<v}\big(\max(v,-j)-\max(u,-j)\big)\,\omega_{\max(u,-j)}^{n-1}\wedge\omega$$

$$+\limsup_{j\to\infty}\int\limits_{\{u\leq -s\}\cap\{u<v\}}\big(\max(v,-j)-\max(u,-j)\big)\,\omega_{\max(u,-j)}^{n-1}\wedge\omega$$

$$=\int\limits_{-s<u<v}(v-u)\,\omega_u^{n-1}\wedge\omega+\limsup_{j\to\infty}\int\limits_{\{u\leq -s\}\cap\{u<v\}}\big(\max(v,-j)-\max(u,-j)\big)\,\omega_{\max(u,-j)}^{n-1}\wedge\omega$$

for all $s>1$. Since $\big(-\max(v,-j)\big)\,\omega_{\max(u,-j)}^{n-1}\wedge\omega<\big(-\max(u,-j)\big)\,\omega_{\max(u,-j)}^{n-1}\wedge\omega\ll Cap_\omega$ in the set $\{u<v\}$ uniformly for all j, letting $s\to\infty$ we get the required inequality and the proof of Lemma 1 is complete.

Theorem 3. *Let $u_0\in\mathcal{F}(X,\omega)$. If $u\in PSH^{-1}(X,\omega)$ satisfies $u\geq u_0$ in X then $u\in\mathcal{F}(X,\omega)$. Moreover, we have that $(-u)\,\omega_u^{n-1}\wedge\omega\ll Cap_\omega$ on X uniformly for all $u\in PSH^{-1}(X,\omega)$ with $u\geq u_0$ in X.*

Proof. Given $k\geq 1$ and $j\geq 1$. Write $u_j=\max(u,-j)$. Then $u_j/3\in\mathcal{F}(X,\omega)$ and by Lemma 1 we have $\int_{u_j<-k}(-u_j)\omega_{u_j}^{n-1}\wedge\omega\leq 2\int_{u_j<-k}(-k/2-u_j)\omega_{u_j}^{n-1}\wedge\omega\leq 3^{n-1}\,2\int_{u_j<-k/2}(-k/2-u_j)\omega_{\frac{1}{3}u_j}^{n-1}\wedge\omega\leq 3^n\int_{u_0<u_j/3-k/3}(u_j/3-k/3-u_0)\,\omega_{\frac{1}{3}u_j}^{n-1}\wedge\omega\leq 3^n\int_{u_0<u_j/3-k/3}(u_j/3-k/3-u_0)\,\omega_{u_0}^{n-1}\wedge\omega\leq 3^n\int_{u_0<-k/3}(-u_0)\,\omega_{u_0}^{n-1}\wedge\omega$. Thus, by $(-u_0)\,\omega_{u_0}^{n-1}\wedge\omega\ll Cap_\omega$ in X we obtain that $(-u_j)\omega_{u_j}^{n-1}\wedge\omega\ll Cap_\omega$ in X uniformly for all j, which yields that $u\in\mathcal{F}(X,\omega)$. Moreover, for all $k\geq 1$, $t\geq 1$ and $u\in PSH^{-1}(X,\omega)$ with $u\geq u_0$, we have $\int_{\max(u,-t)<-k}(-u)\omega_u^{n-1}\wedge\omega\leq\limsup_{j\to\infty}\int_{\max(u,-t)<-k}(-u_j)\,\omega_{u_j}^{n-1}\wedge\omega\leq\limsup_{j\to\infty}\int_{u_j<-k}(-u_j)\omega_{u_j}^{n-1}\wedge\omega\leq 3^n\int_{u_0<-k/3}(-u_0)\,\omega_{u_0}^{n-1}\wedge\omega$. Letting $t\to\infty$, we get $\int_{u<-k}(-u)\omega_u^{n-1}\wedge\omega\leq 3^n\int_{u_0<-k/3}(-u_0)\,\omega_{u_0}^{n-1}\wedge\omega$. Hence, together with $\chi_{\{u>-k-1\}}\,\omega_u^{n-1}\wedge\omega=\chi_{\{u>-k-1\}}\,\omega_{\max(u,-k-1)}^{n-1}\wedge\omega$, we obtain that $(-u)\,\omega_u^{n-1}\wedge\omega\ll Cap_\omega$ on X uniformly for all $u\geq u_0$. The proof of Theorem 3 is complete.

As a direct consequence of Theorem 3 we have

Corollary 1. *Let $u\in\mathcal{F}(X,\omega)$. Then $\max(u,v)\in\mathcal{F}(X,\omega)$ and $t\,u\in\mathcal{F}(X,\omega)$ for all $v\in PSH^{-1}(X,\omega)$ and $0\leq t\leq 1$.*

Now we prove

Theorem 4. *The set $\mathcal{F}(X,\omega)$ is convex, that is, for any u, $v \in \mathcal{F}(X,\omega)$ and $0 \le t \le 1$ we have that* $t\,u + (1-t)\,v \in \mathcal{F}(X,\omega)$.

Proof. Given u, $v \in \mathcal{F}(X,\omega)$. Then $u/2 + v/2 \in PSH^{-1}(X,\omega)$. We only need to prove that $u/2 + v/2 \in \mathcal{F}(X,\omega)$. From Corollary 1 it turns out that $u/2 \in \mathcal{F}(X,\omega)$ and $v/2 \in \mathcal{F}(X,\omega)$. Then $\omega^{n-1}_{\max(u/2,-j)+\max(v/2,-j)} \wedge \omega = 1/2^{n-1}\left(\omega_{\max(u,-2j)} + \omega_{\max(v,-2j)}\right)^{n-1} \wedge \omega \le n!/2^{n-1} \sum\limits_{l=0}^{n-1} \omega^l_{\max(u,-2j)} \wedge \omega^{n-1-l}_{\max(v,-2j)} \wedge \omega$. Write $u_{2j} = \max(u,-2j)$ and $v_{2j} = \max(v,-2j)$. For all $j \ge k \ge 1$ and $0 \le l \le n-1$ we have

$$\int\limits_{u \le -k} \omega^l_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega = 1/k \int\limits_{u \le -k} \big(-\max(u,-k)\big)\,\omega^l_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega$$

$$\le 1/k \int\limits_X (-u_{2j})\,\omega^l_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega \le 1/k \int\limits_{u_{2j} \le v_{2j}} (-u_{2j})\,\omega^l_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega$$

$$+1/k \int\limits_{u_{2j} > v_{2j}} (-v_{2j})\,\omega^l_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega.$$

From Lemma 1 it follows that $\int_{u_{2j} \le v_{2j}} (-u_{2j})\,\omega^l_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega \le 2 \int_{u_{2j} \le v_{2j}} \big(v_{2j}/2 - u_{2j}\big)\,\omega^l_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega \le 2^{n-l} \int_{u_{2j} < v_{2j}/2} \big(v_{2j}/2 - u_{2j}\big)\,\omega^l_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}/2} \wedge \omega \le 2^{n-l} \int_{u_{2j} < v_{2j}/2} \big(v_{2j}/2 - u_{2j}\big)\,\omega^{n-1}_{u_{2j}} \wedge \omega \le 2^{n-l} \sup\limits_j \int_X (-u_{2j})\,\omega^{n-1}_{u_{2j}} \wedge \omega < \infty$. Similarly, we have $\int_{u_{2j} > v_{2j}} (-v_{2j})\,\omega^l_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega \le 2^{l+1} \sup\limits_j \int_X (-v_{2j})\,\omega^{n-1}_{v_{2j}} \wedge \omega < \infty$. Hence we have proved that there exists a constant $A > 0$ such that $\int_{\{u \le -k\} \cup \{v \le -k\}} \omega^{n-1}_{\max(u/2,-j)+\max(v/2,-j)} \wedge \omega \le A/k$ for all $j \ge k \ge 1$. Thus, for $j \ge 2\,k \ge 1$ we have $\int_{u/2+v/2 \le -k} \omega^{n-1}_{\max(u/2+v/2,-j)} \wedge \omega = \int_X \omega^n - \int_{u/2+v/2 > -k} \omega^{n-1}_{\max(u/2+v/2,-j)} \wedge \omega = \int_X \omega^n - \int_{u/2+v/2 > -k} \omega^{n-1}_{\max(u/2,-j)+\max(v/2,-j)} \wedge \omega = \int_{u/2+v/2 \le -k} \omega^{n-1}_{\max(u/2,-j)+\max(v/2,-j)} \wedge \omega \le A/k$, which implies that $\omega^{n-1}_{\max(u/2+v/2,-j)} \wedge \omega \ll Cap_\omega$ on X uniformly for all j and hence $\omega^{n-1}_{u/2+v/2} \wedge \omega = \lim_{j\to\infty} \omega^{n-1}_{\max(u/2+v/2,-j)} \wedge \omega = \lim_{j\to\infty} \omega^{n-1}_{\max(u/2,-j)+\max(v/2,-j)} \wedge \omega$. It then follows from the lower semi-continuity of $-u/2 - v/2$ that $\int_X (-u/2 - v/2)\,\omega^{n-1}_{u/2+v/2} \wedge \omega \le \limsup_{j\to\infty} \int_X \big(-\max(u/4,-j/2) - \max(v/4,-j/2)\big)\,\omega^{n-1}_{\max(u/2,-j)+\max(v/2,-j)} \wedge \omega < \infty$. By Theorem 1 we have obtained that $u/2 + v/2 \in \mathcal{F}(X,\omega)$, which concludes the proof of Theorem 4.

As consequences we have

Corollary 2. *Let u_0, $u_1, \ldots, u_{n-1} \in \mathcal{F}(X,\omega)$. Then*

$$-u_0\,\omega_{u_1} \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} \wedge \omega \ll Cap_\omega \quad \text{on} \quad X.$$

Proof. Since $(u_0+u_1+\ldots+u_{l-1})/l=(1/l)\,u_{l-1}+(1-1/l)\,(u_0+u_1+\ldots+u_{l-2})/(l-1)$ for $l=2,3\ldots,n$, using the induction principle and Theorem 4 we get that $f:=(u_0+u_1+\ldots+u_{n-1})/n\in\mathcal{F}(X,\omega)$. Hence we have that $-u_0\,\omega_{u_1}\wedge\omega_{u_2}\wedge\ldots\wedge\omega_{u_{n-1}}\wedge\omega\le -n^n f\,\omega_{u_1/n}\wedge\omega_{u_2/n}\wedge\ldots\wedge\omega_{u_{n-1}/n}\wedge\omega\le n^n\,(-f)\,\omega_f^{n-1}\wedge\omega\ll Cap_\omega$ on X, which concludes the proof of Corollary 2.

Using Corollary 2 and following the proof of Lemma 1, we get now a stronger version of Lemma 1.

Corollary 3. *Let u, $v\in\mathcal{F}(X,\omega)$ and $0\le l\le n-1$. Then*

$$\int\limits_{u<v}(v-u)\,\omega_v^l\wedge\omega_u^{n-1-l}\wedge\omega\le\int\limits_{u<v}(v-u)\,\omega_u^{n-1}\wedge\omega.$$

Corollary 4. *Let $u_0\in\mathcal{F}(X,\omega)$. Then*

$$-u_1\,\omega_{u_2}\wedge\omega_{u_3}\wedge\ldots\wedge\omega_{u_n}\wedge\omega\ll Cap_\omega\quad\text{on}\quad X$$

uniformly for all $u_l\in PSH^{-1}(X,\omega)$ with $u_l\ge u_0$ and $l=1,2,\ldots,n$.

Proof. Since $f:=(u_1+u_2+\ldots+u_n)/n\ge u_0$ and $f\in\mathcal{F}(X,\omega)$, by Theorem 3 we get that $-u_1\,\omega_{u_2}\wedge\omega_{u_3}\wedge\ldots\wedge\omega_{u_n}\wedge\omega\le n^n\,(-f)\,\omega_f^{n-1}\wedge\omega\ll Cap_\omega$ on X uniformly for all such functions u_l, which concludes the proof of Corollary 4.

Remark. Corollary 4 implies that a function $u\in PSH^{-1}(X,\omega)$ belongs to $\mathcal{F}(X,\omega)$ if and only if $\big(-\max(u,-j)\big)\,\omega_{\max(u,-j)}^l\wedge\omega^{n-l}\ll Cap_\omega$ on X uniformly for all $j\ge1$ and $0\le l\le n-1$.

3. A Convergence Theorem of the Complex Monge-Ampère Operator

In this section we prove a convergence theorem of the complex Monge-Ampère operator in $\mathcal{F}(X,\omega)$. We divide its proof into several lemmas.

Given u_1, $u_2,\ldots,u_{n-1}\in\mathcal{F}(X,\omega)$. By Corollary 2 the current $\omega_{u_1}\wedge\omega_{u_2}\wedge\ldots\wedge\omega_{u_{n-1}}$ is well defined. Now for any $g\in PSH(X,\omega)\cap L^\infty(X)$, we define the wedge product $\omega_{u_1}\wedge\omega_{u_2}\wedge\ldots\wedge\omega_{u_{n-1}}\wedge\omega_g$ in a natural way:

$$\omega_{u_1}\wedge\omega_{u_2}\wedge\ldots\wedge\omega_{u_{n-1}}\wedge\omega_g:=\omega\wedge\omega_{u_1}\wedge\omega_{u_2}\wedge\ldots\wedge\omega_{u_{n-1}}+dd^c(g\,\omega_{u_1}\wedge\omega_{u_2}\wedge\ldots\wedge\omega_{u_{n-1}}).$$

Then we have

Lemma 2. *Let u_0, $u_1,\ldots,u_{n-1}\in\mathcal{F}(X,\omega)$ and f, $g\in PSH(X,\omega)\cap L^\infty(X)$. Then the following equalities hold.*

(a) $$\int\limits_X(-g)\,dd^cf\wedge\omega_{u_1}\wedge\omega_{u_2}\wedge\ldots\wedge\omega_{u_{n-1}}=\int\limits_X(-f)\,dd^cg\wedge\omega_{u_1}\wedge\omega_{u_2}\wedge\ldots\wedge\omega_{u_{n-1}}.$$

(b) $$\int_X (-g)\, dd^c u_0 \wedge \omega_{u_1} \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} = \int_X (-u_0)\, dd^c g \wedge \omega_{u_1} \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}}.$$

Proof. It is no restriction to assume that $f,\, g \le -2$ in X. Write $T = \omega_{u_1} \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}}$. Take two sequences $f_j,\, g_k \in PSH^{-1}(X, A\omega) \cap C^\infty(X)$ for some $A \ge 1$ such that $f_j \searrow f$ and $g_k \searrow g$ in X, see [D1]. It follows from Dini's theorem and quasicontinuity of quasi-psh functions that $f_j \to f$ in Cap_ω on X. So, using $T \wedge \omega \ll Cap_\omega$, we get $f_j\, T \to f\, T$ and hence $dd^c f_j \wedge T \to dd^c f \wedge T$ weakly in X. Similarly, $dd^c g_k \wedge T \to dd^c g \wedge T$ weakly in X. Thus we have $\int_X (-f_j)\, dd^c g \wedge T = \lim_{k\to\infty} \int_X (-f_j)\, dd^c g_k \wedge T = \lim_{k\to\infty} \int_X (-g_k)\, dd^c f_j \wedge T = \lim_{k\to\infty} \int_X (-g_k)\, (A\omega + dd^c f_j) \wedge T - \lim_{k\to\infty} \int_X (-g_k)\, (A\omega) \wedge T = \int_X (-g)\, dd^c f_j \wedge T$, where the last equality follows from the Lebesgue monotone convergence theorem. Then, by lower semi-continuity of $-g$, we get $\int_X (-f)\, dd^c g \wedge T = \lim_{j\to\infty} \int_X (-f_j)\, dd^c g \wedge T = \lim_{j\to\infty} \int_X (-g) dd^c f_j \wedge T = \lim_{j\to\infty} \int_X (-g)(A\omega + dd^c f_j) \wedge T - \int_X (-g)(A\omega) \wedge T \ge \int_X (-g)\, dd^c f \wedge T$. By symmetry we have abtained equality (a). Let $u_l = \max(u_0, -l)$. By (a) we have $\int_X (-g)\, dd^c u_l \wedge T = \int_X (-u_l)\, dd^c g \wedge T$. It follows from Corollary 2 that $u_0\, T$ is a well-defined current and $u_l\, T \to u_0\, T$ as currents in X. Hence we get $\int_X (-g)\, dd^c u_0 \wedge T \le lim_{l\to\infty} \int_X (-g)\, dd^c u_l \wedge T = lim_{l\to\infty} \int_X (-u_l)\, dd^c g \wedge T = \int_X (-u_0)\, dd^c g \wedge T$. On the other hand, $\int_X (-u_0)\, dd^c g_k \wedge T = lim_{l\to\infty} \int_X (-u_l)\, dd^c g_k \wedge T = lim_{l\to\infty} \int_X (-g_k)\, dd^c u_l \wedge T = \int_X (-g_k)\, dd^c u_0 \wedge T$. Letting $k \to \infty$ we get $\int_X (-u_0)\, dd^c g \wedge T \le \int_X (-g)\, dd^c u_0 \wedge T$. Hence we have proved equality (b) and the proof of Lemma 2 is complete.

Lemma 3. *Let $u \in \mathcal{F}(X, \omega)$ and $g \in PSH(X, \omega) \cap L^\infty(X)$. Then the following statements hold.*

(a) $\omega^{n-1}_{\max(u,-j)} \wedge \omega_g \ll Cap_\omega$ *on X uniformly for all j;*

(b) *For each $f \in PSH(X, \omega) \cap L^\infty(X)$, we have that $f\, \omega^{n-1}_{\max(u,-j)} \wedge \omega_g \longrightarrow f\, \omega^{n-1}_u \wedge \omega_g$ weakly in X as $j \to \infty$;*

(c) $(-u)\, \omega^{n-1}_u \wedge \omega_g \ll Cap_\omega$ *on X.*

Proof. It is no restriction to assume that $g \le -2$ in X. Given $j \ge k \ge 1$. By Lemma 2 we have

$$\int_{u \le -k} \omega^{n-1}_{\max(u,-j)} \wedge \omega_g \le 1/k \int_X \big(-\max(u, -k)\big)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega_g$$

$$= 1/k \int_X \big(-\max(u, -k)\big)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega + 1/k \int_X (-g)\, \omega^{n-1}_{\max(u,-j)} \wedge dd^c \max(u, -k)$$

$$\le 1/k \int_X \big(-\max(u, -j)\big)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega + 1/k \int_X (-g)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega_{\max(u,-k)}$$

$$\le 1/k\ \sup_j \int_X \big(-\max(u, -j)\big)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega + 1/k\ \sup_X |g| \int_X \omega^n.$$

Given a Borel set $E \subset X$. By Proposition 4.2 in [BT1] for bounded quasi-psh functions, we get that $\int_E \omega^{n-1}_{\max(u,-j)} \wedge \omega_g \le \int_{u \le k} \omega^{n-1}_{\max(u,-j)} \wedge \omega_g + \int_E \omega^{n-1}_{\max(u,-k)} \wedge \omega_g$ for all $j \ge k \ge 1$, which implies (a).

To prove (b), we prove first that $\omega^{n-1}_{\max(u,-j)} \wedge \omega_g \longrightarrow \omega_u^{n-1} \wedge \omega_g$ weakly in X as $j \to \infty$. Given a smooth function ψ. Multiplying a small positive constant if necessary, we can assume $\psi \in PSH(X,\omega) \cap C^\infty(X)$. Then we have $\int_X \psi\, \omega^{n-1}_{\max(u,-j)} \wedge \omega_g - \int_X \psi\, \omega_u^{n-1} \wedge \omega_g = \int_X \psi \left(\omega^{n-1}_{\max(u,-j)} \wedge \omega - \omega_u^{n-1} \wedge \omega\right) + \int_X g \left(\omega^{n-1}_{\max(u,-j)} - \omega_u^{n-1}\right) \wedge dd^c\psi$, where by Proposition 1 the first term on the right hand side tends to zero as $j \to \infty$. Take a sequence $g_k \in PSH^{-1}(X, A\omega) \cap C^\infty(X)$ for some $A \ge 1$ such that $g_k \searrow g$ in X, see [D1]. Write the second term as

$$\int_X g_k \left(\omega^{n-1}_{\max(u,-j)} - \omega_u^{n-1}\right) \wedge dd^c\psi + \int_X (g - g_k) \left(\omega^{n-1}_{\max(u,-j)} - \omega_u^{n-1}\right) \wedge dd^c\psi := B_{k,j} + C_{k,j}.$$

By the smoothness of ψ we have that $\left(\omega^{n-1}_{\max(u,-j)} + \omega_u^{n-1}\right) \wedge \omega_\psi \ll Cap_\omega$ on X uniformly for all j. Since $g_k \to g$ in Cap_ω on X, we get that $C_{k,j} \to 0$ as $k \to \infty$ uniformly for all j. Then for each fixed k, $B_{k,j} \to 0$ as $j \to \infty$. Hence we have proved that $\omega^{n-1}_{\max(u,-j)} \wedge \omega_g \longrightarrow \omega_u^{n-1} \wedge \omega_g$ weakly in X as $j \to \infty$. Together with (a), we get $\omega_u^{n-1} \wedge \omega_g \ll Cap_\omega$ on X, see the proof of Proposition 1. Now for $f \in PSH(X,\omega) \cap L^\infty(X)$, we take a sequence $f_k \in PSH(X, A\omega) \cap C^\infty(X)$ for some $A \ge 1$ such that $f_k \searrow f$ in X. Write $f\, \omega^{n-1}_{\max(u,-j)} \wedge \omega_g - f\, \omega_u^{n-1} \wedge \omega_g = (f - f_k) \left(\omega^{n-1}_{\max(u,-j)} \wedge \omega_g - \omega_u^{n-1} \wedge \omega_g\right) + f_k \left(\omega^{n-1}_{\max(u,-j)} \wedge \omega_g - \omega_u^{n-1} \wedge \omega_g\right)$, where for each fixed k the second term on the right hand side tends to zero weakly as $j \to \infty$. Using (a) and $\omega_u^{n-1} \wedge \omega_g \ll Cap_\omega$, we get that the first term converges weakly to zero uniformly for all j as $k \to \infty$. Thus we have obtained (b).

Finally, by the lower semi-continuity of $-u$, for any $k \ge 1$ we obtain $\int_X \left(-\max(u,-k)\right) \omega_u^{n-1} \wedge \omega_g \le \limsup_{j\to\infty} \int_X \left(-\max(u,-k)\right) \omega^{n-1}_{\max(u,-j)} \wedge \omega_g \le \sup_j \int_X \left(-\max(u,-j)\right) \omega^{n-1}_{\max(u,-j)} \wedge \omega + \sup_X |g| \int_X \omega^n < \infty$, which yields $u \in L^1(X, \omega_u^{n-1} \wedge \omega_g)$. Thus we have that $(-u)\, \omega_u^{n-1} \wedge \omega_g \ll \omega_u^{n-1} \wedge \omega_g \ll Cap_\omega$ on X. The proof of Lemma 3 is complete.

Lemma 4. *Let $u_0, u_1, \dots, u_{n-1} \in \mathcal{F}(X,\omega)$ and $g \in PSH(X,\omega) \cap L^\infty(X)$. Suppose that a sequence $u_j \in PSH^{-1}(X,\omega)$ decreases to u_1 in X. Then the following statements hold.*

(a) $(-u_0)\, \omega_{u_1} \wedge \omega_{u_2} \wedge \dots \wedge \omega_{u_{n-1}} \wedge \omega_g \ll Cap_\omega$ *on X;*

(b) *For each $f \in PSH(X,\omega) \cap L^\infty(X)$, we have that $f\, \omega_{u_j} \wedge \omega_{u_2} \wedge \dots \wedge \omega_{u_{n-1}} \wedge \omega_g \longrightarrow f\, \omega_{u_1} \wedge \omega_{u_2} \wedge \dots \wedge \omega_{u_{n-1}} \wedge \omega_g$ weakly in X as $j \to \infty$;*

(c) $\omega_{u_j} \wedge \omega_{u_2} \wedge \omega_{u_3} \wedge \dots \wedge \omega_{u_{n-1}} \wedge \omega_g \ll Cap_\omega$ *on X uniformly for all j.*

Proof. Since $(u_0 + u_1 + \dots + u_{n-1})/n \in \mathcal{F}(X,\omega)$, assertion (a) follows directly from (c) of Lemma 3. Now we prove (b). Given a smooth function ψ in X. We assume without loss of generality that $0 \le f,\ \psi \in PSH(X,\omega) \cap L^\infty(X)$. Observe that $\varepsilon h^2 \in PSH(X,\omega)$ if h is a bounded positive quasi-psh function in X and the constant ε satisfies $\max_X h \le 1/(2\varepsilon)$.

Hence, applying the quality $\frac{\psi f}{2} = (\frac{\psi+f}{2})^2 - (\frac{\psi}{2})^2 - (\frac{f}{2})^2$, we can assume that $h := \psi f$ is a bounded quasi-psh function in X. By Lemma 2, for each $k \geq 1$ we get

$$\Big|\int\limits_X \psi f \, \omega_{u_j} \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} \wedge \omega_g - \int\limits_X \psi f \, \omega_{u_1} \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} \wedge \omega_g\Big|$$

$$= \Big|\int\limits_X (u_j - u_1) \, dd^c h \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} \wedge \omega_g\Big| \leq \int\limits_X |u_j - u_1| \, (\omega_h + \omega) \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} \wedge \omega_g$$

$$\leq \int\limits_{u_1 < -k} |u_1| \, (\omega_h + \omega) \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} \wedge \omega_g$$

$$+ \int\limits_X |\max(u_j, -k) - \max(u_1, -k)| \, (\omega_h + \omega) \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} \wedge \omega_g,$$

where by (a) the first term on the right hand side tends to zero as $k \to \infty$. For each fixed k, since $\max(u_j, -k) \to \max(u_1, -k)$ in Cap_ω on X as $j \to \infty$, we get that the second term converges to zero as $j \to \infty$. Hence we have obtained (b).

By (a) and Theorem 3.2 in [BT1], assertion (c) follows from the property: for any hyperconvex subset $\Omega \subset\subset X$ with $dd^c \phi = \omega$ and $\phi = 0$ on $\partial\Omega$ and any $h \in PSH(\Omega) \cap L^\infty(\Omega)$, we have that $h \, \omega_{u_j} \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} \wedge \omega_g \longrightarrow h \, \omega_{u_1} \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} \wedge \omega_g$ weakly in Ω as $j \to \infty$. To prove this property, for each $\psi \in C_0^\infty(\Omega)$, we take a constant $\varepsilon > 0$ such that $\varepsilon \, (h - \sup\limits_\Omega h - 1) > \phi$ on $supp \, \psi$, and $\varepsilon \, (h - \sup\limits_\Omega h - 1) < \phi$ near $\partial\Omega$. Set

$$f = \begin{cases} \max\big(\varepsilon \, (h - \sup\limits_\Omega h - 1), \phi\big) - \phi, & \text{in } \Omega; \\ 0, & \text{in } X \setminus \Omega. \end{cases}.$$

Then $f \in PSH(X, \omega) \cap L^\infty(X)$ and $\psi \, h = \varepsilon^{-1} \psi \phi + \varepsilon^{-1} \psi f + \psi \sup\limits_\Omega h + \psi$. Hence, by the smoothness of ϕ and (b), we get that $h \, \omega_{u_j} \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} \wedge \omega_g \longrightarrow h \, \omega_{u_1} \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} \wedge \omega_g$ weakly in Ω as $j \to \infty$. Therefore, we have proved (c) and the proof of Lemma 4 is complete.

Lemma 5. *Let $u_0, u_1, u_2, \ldots, u_{n-1} \in \mathcal{F}(X, \omega)$ and $g \in PSH(X, \omega) \cap L^\infty(X)$. Then for almost all constants $1 \leq k < \infty$,*

$$\int\limits_{u_1 < -k} (-k - u_1) \, dd^c u_0 \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} \wedge \omega_g \leq \int\limits_{u_1 < -k} (-u_0) \, dd^c u_1 \wedge \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} \wedge \omega_g.$$

Proof. Write $T = \omega_{u_2} \wedge \ldots \wedge \omega_{u_{n-1}} \wedge \omega_g$. Assume first that $0 \geq u_0, u_1 \in PSH(X, A\omega) \cap C^\infty(X)$ with $A \geq 1$. Given $\varepsilon > 0$ and $k \geq 1$. Since $\max(u_1 + \varepsilon, -k) = u_1 + \varepsilon$ near $\partial\{u_1 < -k\}$ if it is not empty, we have that $\int_{u_1 < -k} (-k - u_1) \, dd^c u_0 \wedge T = \lim_{\varepsilon \searrow 0} \int_{u_1 < -k} \big(\max(u_1 +$

$\varepsilon,-k) - u_1 - \varepsilon\big)\, dd^c u_0 \wedge T = \lim_{\varepsilon\searrow 0}\int_{u_1<-k} u_0\, dd^c\big(\max(u_1+\varepsilon,-k) - u_1 - \varepsilon\big) \wedge T = \int_{u_1<-k}(-u_0)\, dd^c u_1 \wedge T + \lim_{\varepsilon\searrow 0}\int_{u_1<-k} u_0\, dd^c \max(u_1+\varepsilon,-k) \wedge T$. Since $\max(u_1 + \varepsilon,-k)\,T \longrightarrow \max(u_1,-k)\,T$ weakly in X as $\varepsilon \searrow 0$, we have $\big(A\omega + dd^c\max(u_1+\varepsilon,-k)\big) \wedge T \longrightarrow \big(A\omega + dd^c\max(u_1,-k)\big)\wedge T$ weakly as $\varepsilon\searrow 0$. From the upper semi-continuity of $u_0 \le 0$ in the open set $\{u_1<-k\}$, it turns out that $\lim_{\varepsilon\searrow 0}\int_{u_1<-k} u_0\, dd^c\max(u_1+\varepsilon,-k)\wedge T = \lim_{\varepsilon\searrow 0}\int_{u_1<-k} u_0 \,\big[\big(A\omega + dd^c\max(u_1+\varepsilon,-k)\big) - A\omega\big]\wedge T \le \int_{u_1<-k} u_0\, dd^c\max(u_1,-k)\big)\wedge T = 0$. Hence we get $\int_{u_1<-k}(-k-u_1)\, dd^c u_0\wedge T \le \int_{u_1<-k}(-u_0)\, dd^c u_1 \wedge T$ for all $k\ge 1$ in the case of $0 \ge u_0,\ u_1 \in PSH(X, A\omega)\cap C^\infty(X)$.

Secondly, assume that $u_0,\ u_1 \in \mathcal{F}(X,\omega)\cap L^\infty(X)$. By [D1] there exist negative functions $u_{0t},\ u_{1s} \in PSH(X,A\omega)\cap C^\infty(X)$ with some $A\ge 1$ such that $u_{0t}\searrow u_0$ and $u_{1s}\searrow u_1$ in X. Since $\int_{u_1\le -k}\big(\omega_{u_1}+\omega\big)\wedge T$ is an increasing function of k and hence continuous almost everywhere with respect to the Lebesgue measure, we have that $\int_{u_1=-k}\big(\omega_{u_1}+\omega\big)\wedge T = 0$ holds for almost all k in $[1,\infty)$. Given such a constant k. By Fatou lemma and the lower semi-continuity of $-u_{1s}$, we get that $\int_{u_1<-k}(-k-u_1)\, dd^c u_0\wedge T = \int_{u_1<-k}(-k - u_1)\,(A\omega + dd^c u_0)\wedge T - A\int_{u_1<-k}(-k-u_1)\,\omega\wedge T \le \liminf_{s\to\infty}\int_{u_{1s}<-k}(-k-u_{1s})\,(A\omega + dd^c u_0)\wedge T - A\int_{u_1<-k}(-k-u_1)\,\omega\wedge T \le \liminf_{s\to\infty}\limsup_{t\to\infty}\int_{u_{1s}<-k}(-k-u_{1s})\,(A\omega + dd^c u_{0t})\wedge T - \liminf_{s\to\infty} A\int_{u_1<-k}(-k-u_{1s})\,\omega\wedge T = \liminf_{s\to\infty}\limsup_{t\to\infty}\int_{u_{1s}<-k}(-k-u_{1s})\, dd^c u_{0t}\wedge T - A\liminf_{s\to\infty}\int_{u_{1s}\ge -k > u_1}(-k-u_{1s})\,\omega\wedge T$. Given $\delta>0$, we have that $\big|\int_{u_{1s}\ge -k>u_1}(-k-u_{1s})\,\omega\wedge T\big| \le \delta\int_X \omega\wedge T + \int_{u_{1s}-u_1\ge\delta}(-u_1)\,\omega\wedge T \longrightarrow \delta\int_X\omega\wedge T$ as $s\to\infty$, since $u_{1s}\to u_1$ in Cap_ω and $(-u_1)\,\omega\wedge T \ll Cap_\omega$ on X. Hence we have $\int_{u_1<-k}(-k-u_1)\, dd^c u_0 \wedge T \le \liminf_{s\to\infty}\limsup_{t\to\infty}\int_{u_{1s}<-k}(-k-u_{1s})\, dd^c u_{0t}\wedge T \le \liminf_{s\to\infty}\limsup_{t\to\infty}\int_{u_{1s}<-k}(-u_{0t})\, dd^c u_{1s}\wedge T = \liminf_{s\to\infty}\int_{u_{1s}<-k}(-u_0)\, dd^c u_{1s}\wedge T \le \liminf_{s\to\infty}\int_{u_1\le -k}(-u_0)\,(A\omega + dd^c u_{1s})\wedge T - A\liminf_{s\to\infty}\int_{u_{1s}<-k}(-u_0)\,\omega\wedge T = \liminf_{s\to\infty}\int_{u_1\le -k}(-u_0)\,(A\omega+dd^c u_{1s})\wedge T - A\int_{u_1\le -k}(-u_0)\,\omega\wedge T$. By Lemma 4 and quasicontinuity of quasi-psh functions, it is no restriction to assume that $\{u_1\le -k\}$ is a closed set and hence the last limit inferior does not exceed $\int_{u_1\le -k}(-u_0)\,(A\omega + dd^c u_1)\wedge T$. So we have obtained $\int_{u_1<-k}(-k-u_1)\, dd^c u_0\wedge T \le \int_{u_1<-k}(-u_0)\, dd^c u_1\wedge T$ for all $u_0,\ u_1\in \mathcal{F}(X,\omega)\cap L^\infty(X)$ and almost all k in $[1,\infty)$.

Finally, let $u_0,\ u_1 \in \mathcal{F}(X,\omega)$. For almost all constants k in $[1,\infty)$ we have that $\int_{u_1=-k}\big(\omega_{u_1}+\omega\big)\wedge T = 0$ and $\int_{\max(u_1,-s)<-k}\big(-k-\max(u_1,-s)\big)\, dd^c\max(u_0,-t)\wedge T \le \int_{\max(u_1,-s)<-k}\big(-\max(u_0,-t)\big)\, dd^c\max(u_1,-s)\wedge T$ for all integers $s,\ t\ge 1$. Letting $s\to\infty$ and applying the same proof as above, we have $\int_{u_1<-k}(-k_j-u_1)\, dd^c\max(u_0,-t)\wedge T \le \int_{u_1<-k}\big(-\max(u_0,-t)\big)\, dd^c u_1\wedge T$ and then letting $t\to\infty$ we get the required inequality. The proof of Lemma 5 is complete.

Lemma 6. *Let $u_0\in\mathcal{F}(X,\omega)$ and $g\in PSH(X,\omega)\cap L^\infty(X)$. Then*

$$\int\limits_{u<-k} (-u)\,\omega_u^{n-1}\wedge\omega_g \longrightarrow 0, \qquad \text{as} \quad k\to\infty,$$

uniformly for all $u\in PSH^{-1}(X,\omega)$ with $u\ge u_0$ in X.

Proof. Given $u \in PSH^{-1}(X,\omega)$ with $u \geq u_0$. Take a sequence $1 \leq k_1 \leq k_2 \leq \ldots \leq k_j \to \infty$ such that Lemma 5 holds for the functions u and u_0 when $k = k_j/2^i$, where $i = 1, \ldots, n-1$ and $j = 1, 2, \ldots$. Hence we have

$$\int\limits_{u<-k_j} (-u)\,\omega_u^{n-1} \wedge \omega_g \leq \int\limits_{u_0<-k_j} (-u_0)\,\omega_u^{n-1} \wedge \omega_g \leq 2 \int\limits_{u_0<-k_j} (-k_j/2 - u_0)\,\omega_u^{n-1} \wedge \omega_g$$

$$\leq 2 \int\limits_{u_0<-k_j/2} (-k_j/2 - u_0)\,\omega \wedge \omega_u^{n-2} \wedge \omega_g + 2 \int\limits_{u_0<-k_j/2} (-k_j/2 - u_0)\,dd^c u \wedge \omega_u^{n-2} \wedge \omega_g$$

$$\leq 2 \int\limits_{u_0<-k_j/2} (-k_j/2 - u_0)\,\omega \wedge \omega_u^{n-2} \wedge \omega_g + 2 \int\limits_{u_0<-k_j/2} (-u)\,dd^c u_0 \wedge \omega_u^{n-2} \wedge \omega_g$$

$$\leq 2 \int\limits_{u_0<-k_j/2} (-u_0)\,\omega \wedge \omega_u^{n-2} \wedge \omega_g + 2 \int\limits_{u_0<-k_j/2} (-u_0)\,\omega_{u_0} \wedge \omega_u^{n-2} \wedge \omega_g$$

$$= 2 \int\limits_{u_0<-k_j/2} (-u_0)\,(\omega + \omega_{u_0}) \wedge \omega_u^{n-2} \wedge \omega_g \leq 2^2 \int\limits_{u_0<-k_j/2^2} (-u_0)\,(\omega + \omega_{u_0})^2 \wedge \omega_u^{n-3} \wedge \omega_g$$

$$\leq \ldots \leq 2^{n-1} \int\limits_{u_0<-k_j/2^{n-1}} (-u_0)\,(\omega + \omega_{u_0})^{n-1} \wedge \omega_g,$$

which, by Lemma 4 and the equality $(\omega + \omega_{u_0})^{n-1} = \sum\limits_{l=0}^{n-1} \binom{n-1}{l}\,\omega^l \wedge \omega_{u_0}^{n-1-l}$, tends to zero as $j \to \infty$. This concludes the proof of Lemma 6.

We are now in a position to prove the convergence theorem.

Theorem 5.(Convergence Theorem) *Let $0 \leq p < \infty$. Suppose that $0 \geq g \in PSH(X,\omega) \cap L^\infty(X)$ and $u_0 \in \mathcal{F}(X,\omega)$. If u_j, $u \in PSH^{-1}(X,\omega)$ are such that $u_j \to u$ in Cap_ω on X and $u_j \geq u_0$, then $(-g)^p\,\omega_{u_j}^n \to (-g)^p\,\omega_u^n$ weakly in X.*

Proof. Given $k \geq 1$. Write

$$(-g)^p\,\omega_{u_j}^n - (-g)^p\,\omega_u^n = (-g)^p\left(\omega_{u_j}^n - \omega_{\max(u_j,-k)}^n\right) + (-g)^p\left(\omega_{\max(u_j,-k)}^n - \omega_{\max(u,-k)}^n\right)$$

$$+(-g)^p\left(\omega_{\max(u,-k)}^n - \omega_u^n\right) := A_{k,j} + B_{k,j} + C_k.$$

For each fixed k, by Theorem 1 in [X3] we have that $B_{k,j} \to 0$ weakly in X as $j \to \infty$. Given a smooth function ψ in X. Following the proof of Theorem 1 in [X3], we can assume that $\psi\,(-g)^p$ is the sum of finite terms of form $\pm f$, where f are bounded quasi-psh functions in X. For such a function f, by Lemma 2 we get

$$\left|\int_X f\left(\omega_{u_j}^n - \omega_{\max(u_j,-k)}^n\right)\right| = \left|\int_X (u_j - \max(u_j,-k))\,dd^c f \wedge \sum_{l=0}^{n-1} \omega_{u_j}^l \wedge \omega_{\max(u_j,-k)}^{n-1-l}\right|$$

$$= \Big| \int_{u_j < -k} (u_j + k)\, dd^c f \wedge \sum_{l=0}^{n-1} \omega_{u_j}^l \wedge \omega_{\max(u_j,-k)}^{n-1-l} \Big| \le \int_{u_j<-k} (-u_j)\,(\omega_f + \omega) \wedge \omega_{u_j}^{n-1},$$

which by Lemma 6 tends to zero uniformly for all j as $k \to \infty$. Hence, $A_{k,j} \to 0$ uniformly for all j as $k \to \infty$. Similarly, we have that $C_k \to 0$ weakly as $k \to \infty$. Therefore, we have obtained that $(-g)^p\, \omega_{u_j}^n \to (-g)^p\, \omega_u^n$ weakly and the proof of Theorem 5 is complete.

Applying Dini's theorem and quasicontinuity of quasi-psh functions, we get the following consequence.

Corollary 5. *Let $0 \le p < \infty$ and $0 \ge g \in PSH(X,\omega) \cap L^\infty(X)$. If u_j, $u \in \mathcal{F}(X,\omega)$ are such that $u_j \searrow u$ or $u_j \nearrow u$ in X, then $(-g)^p\, \omega_{u_j}^n \to (-g)^p\, \omega_u^n$ weakly in X.*

Corollary 6. *Let u, $v \in \mathcal{F}(X,\omega)$. Then*

$$\chi_{\{u>v\}}\, \omega_{\max(u,v)}^n = \chi_{\{u>v\}}\, \omega_u^n.$$

Proof. This proof is similar to the proof of Theorem 4.1 in [KH]. Given a constant $k \ge 0$. Write $u_j = \max(u, -j)$. By Proposition 4.2 in [BT1] we have that $\max(u_j + k, 0)\, \omega_{\max(u_j,-k)}^n = \max(u_j + k, 0)\, \omega_{u_j}^n$ for all j. Using $\max(u_j + k, 0) \ge \max(u + k, 0) \ge 0$, we get $\max(u + k, 0)\, \omega_{\max(u_j,-k)}^n = \max(u + k, 0)\, \omega_{u_j}^n$. Letting $j \to \infty$ and applying Theorem 5, we get $\max(u + k, 0)\, \omega_{\max(u,-k)}^n = \max(u + k, 0)\, \omega_u^n$. Hence we have obtained that $\chi_{\{u>-k\}}\, \omega_{\max(u,-k)}^n = \chi_{\{u>-k\}}\, \omega_u^n$ holds for any $u \in \mathcal{F}(X,\omega)$ and $k \ge 0$. Therefore, $\omega_{\max(u,v)}^n = \omega_{\max(u,v,-k)}^n$ and $\omega_u^n = \omega_{\max(u,-k)}^n$ on each set $\{u > -k > v\}$ with a rational number $k \ge 0$. But $\omega_{\max(u,v,-k)}^n = \omega_{\max(u,-k)}^n$ on the open set $\{-k > v\}$ and hence $\chi_{\{u>-k>v\}}\, \omega_{\max(u,v)}^n = \chi_{\{u>-k>v\}}\, \omega_u^n$, which implies the required equality. The proof of Corollary 6 is complete.

Corollary 7. *Let u, $v \in \mathcal{F}(X,\omega)$. Then*

$$\omega_{\max(u,v)}^n \ge \chi_{\{u \ge v \text{ and } u \ne -\infty\}}\, \omega_u^n + \chi_{\{u<v\}}\, \omega_v^n.$$

Proof. Given $\varepsilon > 0$, by Corollary 6 we have $\omega_{\max(u,v-\varepsilon)}^n \ge \chi_{\{u>v-\varepsilon\}}\, \omega_u^n + \chi_{\{u<v-\varepsilon\}}\, \omega_v^n \ge \chi_{\{u \ge v \text{ and } u \ne -\infty\}}\, \omega_u^n + \chi_{\{u<v-\varepsilon\}}\, \omega_v^n$. Letting $\varepsilon \searrow 0$ and using Theorem 5, we obtain the required inequality which concludes the proof.

Remark. Corollary 7 is a generalization of the well known Demailly inequality, see [D2].

Corollary 8. *Let u, $v \in \mathcal{F}(X,\omega)$. Then*

$$\int\limits_{u<v} \omega_v^n \le \int\limits_{u<v} \omega_u^n + \int\limits_{u=v=-\infty} \omega_u^n.$$

Proof. By Corollary 6 we have $\int_{u<v} \omega_v^n = \int_{u<v} \omega_{\max(u,v)}^n = \int_X \omega^n - \int_{u \ge v} \omega_{\max(u,v)}^n \le \int_X \omega^n - \int_{u>v} \omega_{\max(u,v)}^n = \int_X \omega^n - \int_{u>v} \omega_u^n = \int_{u \le v} \omega_u^n$. Using $\delta\, v$ instead of v and letting $\delta \nearrow 1$, we get the required inequality and the proof is complete.

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